Hamming Graphs

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1 Abstract

In this project we basically aim to analyse the properties of Hamming Graphs.

It is defined in the following way: The Hamming Graphs $G_{n,d}$ = (V_G,E_G) is given by V_G= $\{0,1\}^n$ and two vertices u,v belong to the graph G are linked if Hamming distance d(u,v) between the corresponding the vertices greater than or equal to d.

We have analysed the properties of the Hamming Graphs. Cartesian product of these graphs have been thoroughly studied. We have also analysed the structural properties of tensor product of the Hamming graphs.

Various properties of the graphs like:

1. No. of edges
2. Presence of Eulerian Circuits
3. Chromatic number
4. Connectivity
5. Regularity
6. Diameter

have been studied.

These studies might prove helpful in further research of these kind of graphs.

2 Introduction

Graph theory has been an interesting part of study for both mathematicians as well as the computer scientists. Recent advancements have been made in this field with respect to various types of graphs and their properties have been extensively utilised.

In our paper we have outlined the basic structure of Hamming graphs.This would require basic understanding of various structural aspects of graphs.

2.1 Some Definitions:

- **Eulerian Path:** In graph theory, an Eulerian trail (or Eulerian path) is a trail in a graph which visits every edge exactly once
- **Regular Graph:** In graph theory, a regular graph is a graph where each vertex has the same number of neighbours; i.e. every vertex has the same degree.
- **Chromatic number:** The smallest number of colors needed to color a graph G is called its chromatic number, and is often denoted $\chi(G)$.The colors are given in such a way that non two nodes have same color.
- **Diameter:** The diameter of the graph is the largest number of vertices which must be traversed in order to travel from one vertex to another when paths which backtrack, detour, or loop are excluded from consideration.
- **Representation of node in Hamming Graphs:**Representation of a node in Hamming graphs is asequence of $0'$s & $1'$s for a particular node.

3 Problem Statement

Hamming Graphs $G_{n,d}$=(V_G, E_G) is given by V_G= $\{0,1\}^n$ and two vertices u,v ϵ G are linked if Hamming distance d(u,v) between the corresponding the vertices is greater than or equal to d. These graphs vary on two parameters n and d.

Cartesian product of these huge graphs could reveal certain new properties which could be exploited in various fields. Its difficult to exploit the structural properties of these graphs based on similarities obtained from simple graph and generalising them on a whole. Tensor product of Hamming graphs also have been studied which revealed a lot of properties useful enough for further research on the problem.

4 Methodologies

4.1 Hamming graphs

Hamming Graphs $G_{n,d}$=(V_G, E_G) is given by vertex set V_G= $\{0,1\}^n$ and two vertices u,v ϵ G are linked if Hamming distance d(u,v) between the corresponding the vertices $\geq$ d.

4.2 Examples

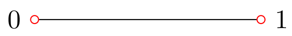

Figure 1: $G_{1,1}$

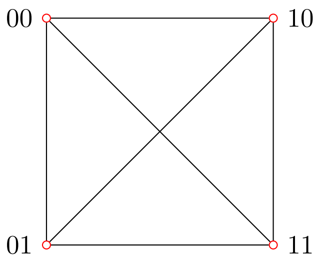

Figure 2: $G_{2,1}$

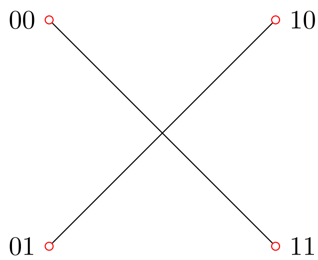

Figure 3: $G_{2,2}$

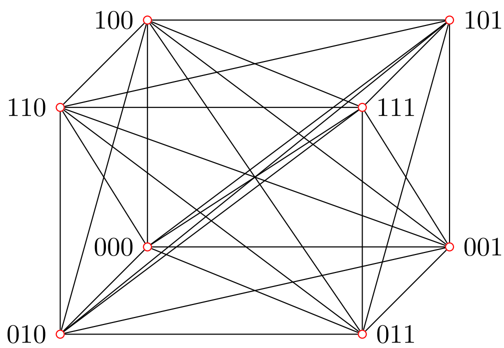

Figure 4: $G_{3,1}$

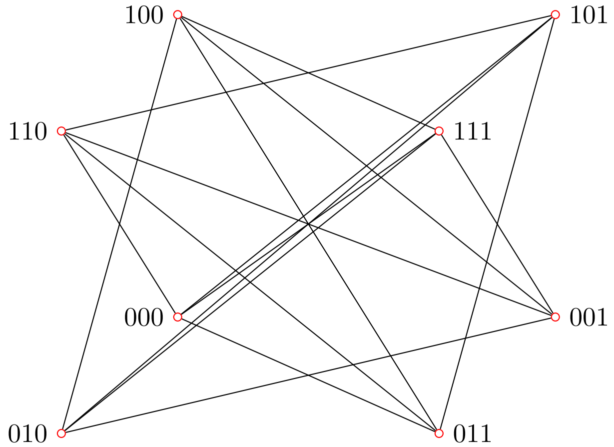

Figure 5: $G_{3,2}$

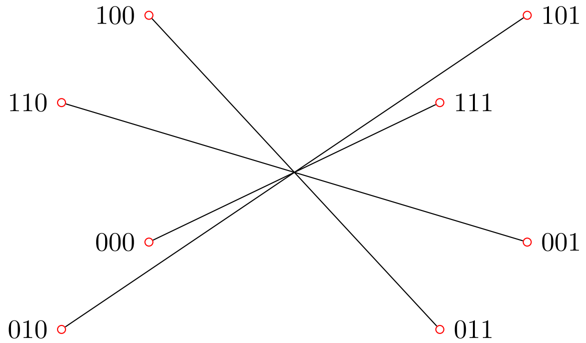

Figure 6: $G_{3,3}$

4.3 Properties of Hamming Graphs

Theorems

1. **Theorem 1:** If there are n_1 points from distance k from a point a,then there are n_1 points at distance k from any other points.

   Proof : There are n bits in the representation of the node of $G_{n,d}$.In order to find a node at a distance k from it we need to select k bits from n bits whose values are to be swapped.This can be possible in ${}^nC_k$ ways.Hence the same formulation applies to any other node in the graph $G_{n,d}$. ■

2. **Theorem 2:** $G_{n,d}$ is a regular graph $\forall$ n,d

   Proof : From Theorem 1 it is evident that for the distance to be minimum d from any vertex or node we need to select d bits in the representation of the node.Applying Theorem 1 to every node we end up in a conclusion that every node has same degree.This implies the fact that $G_{n,d}$ is regular. ■

3. **Theorem 3:** No Eulerian path exists in $G_{n,d}$.

   Proof : For a Eulerian path to exist there has to be 2 nodes of odd degree and other nodes of even degree.But as the graph is regular either all vertices have odd degree or all the vertices have even degree.This evidently proves the fact that $\exists$ no Eulerian path in $G_{n,d}$. ■

4. **Theorem 4:** $\mathrm{R}(G_{n,d}) = 2^n - \sum_{i=0}^{d-1} {}^nC_i$ where R is the regularity of the graph.

   Proof : From Theorem 1 we can select ${}^nC_k$ nodes from 2^n which are at a distance k from a node.For a minimum distance d then our selection runs from $\sum_{i=d}^{n} {}^nC_i$.But from Binomial Theorem we have, $\sum_{i=0}^{n} {}^nC_i = 2^n$.Using this theorem we get $\mathrm{R}(G_{n,d}) = 2^n - \sum_{i=0}^{d-1} nC_i$ ■

5. **Theorem 5:** $\mathrm{N}(G_{n,d}) = \frac{2^n.(2^n - \sum_{i=0}^{d-1} {}^nC_i)}{2}$ where N is the number of edges in the graph.

   Proof : Regularity of a graph gives us the measure of number of edges incident on the node.Therefore on every node we have $2^n - \sum_{i=0}^{d-1} nC_i$ edges incident on it.But there are 2^n nodes in the graph. Hence total number of edges in the graph is $2^n.(2^n - \sum_{i=0}^{d-1} {}^nC_i)$.But during this counting we have calculated every single edge twice.($1 \to 2$ & $2 \to 1$ are equivalent in undirected Hamming graph).So we need to divide the obtained number of edges by 2.Therefore we get number of edges in the graph to be $\frac{2^n.(2^n - \sum_{i=0}^{d-1} nC_i)}{2}$. ■

6. **Theorem 6:** For d=1 Hamming graph $G_{n,d}$ is a complete graph.

   Proof : A complete graph is a graph which is connected to every other node in the graph.For d=1 we are supposed to manipulate only 1bit,only2 bits,...,all bits to obtain the edges.So we will obtain that every vertex is connected to $2^n - 1$ nodes.As there are 2^n nodes and every node is connected to $2^n - 1$ nodes,this proves that it ia complete graph. ■

7. **Theorem 7:** For d=n Hamming graph is a disconnected graph with n parts.

   Proof : For a disconnected graph,$\exists$ two vertices (u,v) such that $\exists$ no path between them in the graph.If d=n then an edge exists between a node to other node whose all bits are different from the given node.This will only result in formation of edges only for nodes whose $\tilde{u}$+v(binary addition) results in (1...n zeros).Hence we would get n disconnected parts. ■

Some Observations:

- Diameter of the Hamming Graph $G_{n,d}$ is d.

5 Cartesian Product of Graphs

Definition: Cartesian product of graphs G and H is defined as follows:the vertex set of G$\circ$H is the Cartesian product V(G)$\times$V(H); and any two vertices (u,u′) and (v,v′) are adjacent in G$\circ$H if and only if either:

- u = v and u′ is adjacent with v′ in H, or

- u′ = v′ and u is adjacent with v in G.

5.1 Examples

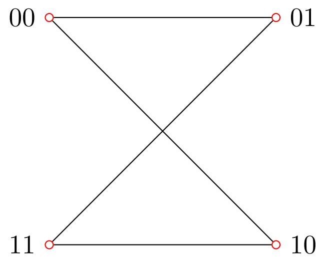

Figure 7: $G_{1,1} \times G_{1,1}$

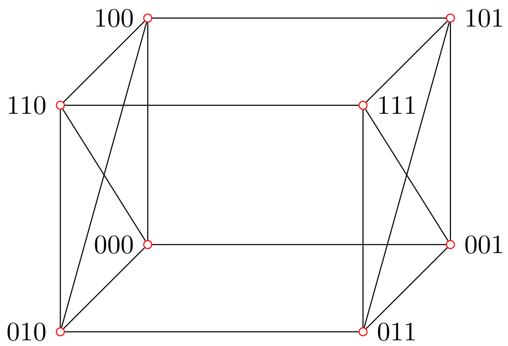

Figure 8: $G_{2,1} \times G_{1,1}$

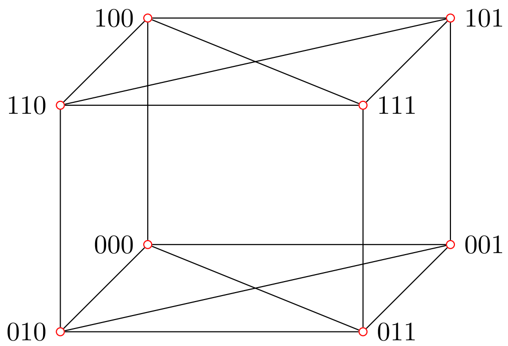

Figure 9: $G_{1,1} \times G_{2,1}$

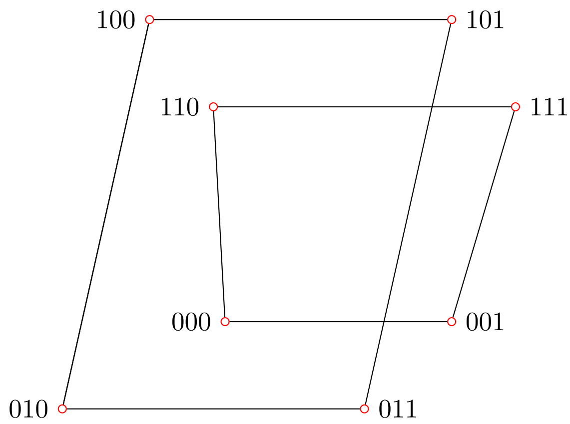

Figure 10: $G_{2,2} \times G_{1,1}$

5.2 Properties of Cartesian product of Hamming Graphs

Theorems:

1. **Theorem 1:** If G_{n_1,d_1} , G_{n_2,d_2} are r_1, r_2 regular; then $G_{n_1,d_1} \times G_{n_2,d_2}$ are $r_1 + r_2$ regular.

   Proof : Given G_{n_1,d_1} , G_{n_2,d_2} are r_1, r_2 regular respectively.When we take take the cartesian product we give a link between a node of one graph to a node of another graph.As Graph G_{n_1,d_1} is r_1 regular and G_{n_2,d_2} is r_2 regular,linking these two nodes would give rise degree of every node to be $r_1 + r_2$.Hence $r_1 + r_2$ ■

2. **Theorem 2:** D(G_{n_1,d_1}) × D(G_{n_2,d_2}) = $d_1 + d_2$ where D is the diameter of the graph.

   Proof : In the cartesian product of Hamming Graphs in order to reach the farthest node in the graph we need to change the bits of the node from where we start one by one and then to reach the farthest vertex we travel d_1.Similarly for H we also observe that we need to travel d_2.Therefore, we travel $d_1 + d_2$ edges which is the diameter of the Cartesian product of graph. ■

3. **Theorem 3:** N($G_{n_1,d_1} \times G_{n_2,d_2}$) = $2^{n_2}.N(G_{n_1,d_1}) + 2^{n_1}.N(G_{n_2,d_2})$

   Proof : Now consider a node (u,v_j) where $v_j's$ are nodes from graph H_{n_2,d_2}.Now edges exist for all $v_j's$.So there exists N(H) edges.If we take all combinations with respect to u we get $2^{n_2}.N(G_{n_1,d_1})$ edges.Applying the same when a node (u_i,v) is taken we get $2^{n_1}.N(G_{n_2,d_2})$ edges.Adding them we get the total number of edges. ■

4. **Theorem 4:** No Eulerian path exists in $B_{n_1,d_1} \times B_{n_2,d_2}$.

   Proof : From Theorem 1 we have proved that $B_{n_1,d_1} \times B_{n_2,d_2}$ is $r_1 + r_2$ regular in nature.For the existence of Eulerian trails it is necessary that no more than two vertices have an odd degree.But here either all nodes have even degree or odd degree.Hence there exists no eulerian trial in the product. ■

5.3 Some Observations:

- Diameter of $G_{n,d} \times G_{n,d}$ is $d_1 + d_2$.
- $G_{n_1,d_1} \times G_{n_2,d_2} \cong G_{n_2,d_2} \times G_{n_1,d_1}$.
- Cartesian Product of two Connected Hamming graphs is connected & disconnected graph with connected/disconnected graph is disconnected.

6 Tensor product of Hamming Graphs

Definition: Tensor product of two Hamming Graphs $G_{n_1,d_1} \otimes G_{n_2,d_2}$ is defined as follows:

- the vertex set of $G_{n_1,d_1} \otimes G_{n_2,d_2}$ is the cartesian product V(G_{n_1,d_1}) × V(G_{n_2,d_2})
- any two vertices (u,v) & (u′,v′) are connected in the tensor product if u is connected to v in G_{n_1,d_1} and u′ is connected to v′ in G_{n_2,d_2}.

6.1 Examples

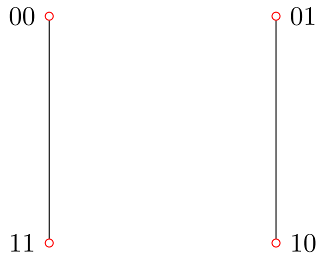

Figure 11: $G_{1,1} \otimes G_{1,1}$

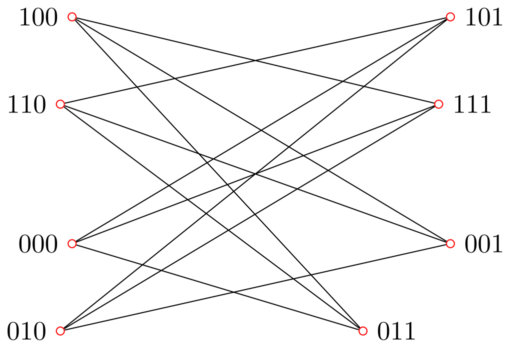

Figure 12: $G_{2,1} \otimes G_{1,1}$

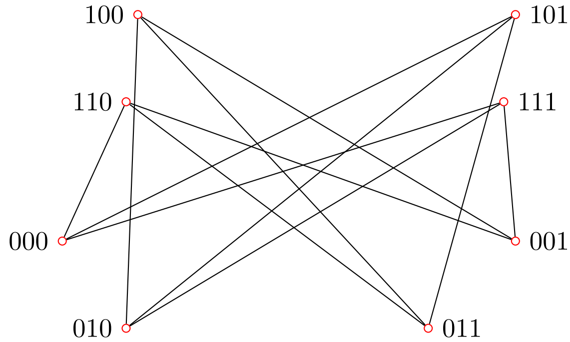

Figure 13: $G_{1,1} \otimes G_{2,1}$

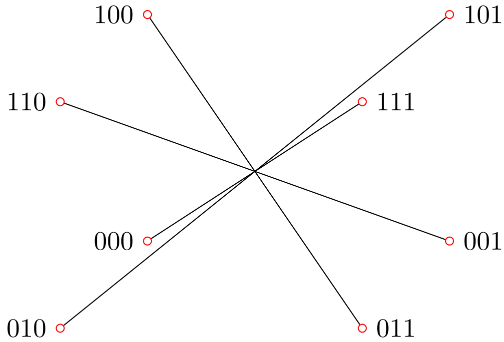

Figure 14: $G_{2,2} \otimes G_{1,1}$

6.2 Properties of Tensor product of Hamming Graphs

Theorems:* [1]

1. **Theorem 1:** The Tensor Product of Hamming Graphs $G_{n_1,d_1} \otimes G_{n_2,d_2}$ results in a Hamming Graph $G_{n_1+n_2,d_1+d_2}$ when $n_1 = d_1$ & $n_2 = d_2$.

   Proof :(The proof is applicable when $n_1 = d_1$ & $n_2 = d_2$) According to the definition of Tensor product the vertex set will include $n_1 + n_2$ bits as it is the cartesian product of vertex sets.From the definition of edge linkings (u, u') are connected if they are connected in G and hence d_1 length.Similarly for (v, v').Hence the minimum Hamming distance becomes $d_1 + d_2$. ■

2. **Theorem 2:** The number of edges in $G_{n_1,d_1} \otimes G_{n_2,d_2}$ is N($G_{n_1,d_1} \otimes G_{n_2,d_2}) = \frac{2^{n_1+n_2}(2^{n_1+n_2} - \sum_{i=0}^{d_1+d_2-1} {}^{n_1+n_2}C_i)}{2}$

   Proof : Regularity of a graph gives us the measure of number of edges incident on the node.Therefore on every node we have $(2^{n_1+n_2} - \sum_{i=0}^{d_1+d_2-1} {}^{n_1+n_2}C_i)$ edges incident on it.But there are $2^{n_1+n_2}$ nodes in the graph.Hence total

[1]* : The Theorems are valid when $n_1 = d_1$ & $n_2 = d_2$

number of edges in the graph is $2^{n_1+n_2}.(2^{n_1+n_2} - \sum_{i=0}^{d_1+d_2-1} {}^{n_1+n_2}C_i)$.But during this counting we have calculated every single edge twice.(1 → 2 & 2 → 1 are equivalent in undirected Hamming graph).So we need to divide the obtained number of edges by 2.Therefore we get number of edges in the graph to be $\frac{2^{n_1+n_2}(2^{n_1+n_2} - \sum_{i=0}^{d_1+d_2-1} {}^{n_1+n_2}C_i)}{2}$. ■

3. **Theorem 3:** $G_{n_1,d_1} \otimes G_{n_2,d_2}$ is a regular graph $\forall\ n_1, n_2, d_1, d_2$

   Proof : In this graph we find matches or links for which u in (u,v) & (u', v') possess a link in G_{n_1,d_1}.Let this be m_1.Similarly we do the same thing for v in (u,v).Let this be m_2.We need to consider the cases where both m_1 & m_2 case intersect as only in that case edge is formed.This will be true for all nodes.Hence the graph is regular. ■

4. **Theorem 4:** R$(G_{n_1,d_1} \otimes G_{n_2,d_2}) = 2^{n_1+n_2}(2^{n_1+n_2} - \sum_{i=0}^{d_1+d_2-1} {}^{n}C_i)$ where R is the regularity of the graph.

   Proof : We can select ${}^{n_1+n_2}C_k$ nodes from $2^{n_1+n_2}$ which are at a distance k from a node.For a minimum distance $d_1 + d_2$ then our selection runs from $\sum_{i=d_1+d_2}^{n_1+n_2} {}^{n_1+n_2}C_i$.But from Binomial Theorem we have, $\sum_{i=0}^{n_1+n_2} {}^{n_1+n_2}C_i = 2^{n_1+n_2}$.Using this theorem we get R$(G_{n_1,d_1} \otimes H_{n_2,d_2})$ $= 2^{n_1+n_2}(2^{n_1+n_2} - \sum_{i=0}^{d_1+d_2-1} {}^{n}C_i)$. ■

6.3 Some Observations:

1. $G_{n_1,d_1} \otimes G_{n_2,d_2} \cong G_{n_2,d_2} \otimes G_{n_1,d_1}$
2. If $(n_1 = d_1\ \&\ n_2 \neq d_2)$ or $(n_1 \neq d_1\ \&\ n_2 = d_2)$,then regularity of $G_{n_1,d_1} \otimes G_{n_2,d_2}$ is equal to $n_1 + n_2 - 1$.

7 Conclusion

Thorough analysis of Hamming graphs gives us a lot of properties which could be exploited in the coding theory for generating codes.We have successfully analysed the structural properties of Hamming graphs with respect properties like no. of edges, chromatic number, regularity etc.We have also studied about the properties of cartesian product of Hamming Graphs and tensor product of Hamming graphs and generalised few theorems.Several examples have presenteed in the document for ease of understanding and verification of theorems.

8 References

1. Richard Hammack, Wilfried Imrich, Sandi Klavar Handbook of Product Graphs, Second Edition Discrete Mathematics and Its Applications 2011.pdf Richard Hammack, Wilfried Imrich, Sandi Klavar Handbook of Product Graphs, Second Edition Discrete Mathematics and Its Applications 2011.pdf